An Overview on Hesitant Fuzzy Information Measures †

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***Abstract*—**Although the concept of fuzzy set (FS) has been widely and successfully applied in many different areas to model some types of uncertainty, the limitation of this concept is still more serious in case of dealing with imprecise and vague information when different sources of vagueness appear simultaneously. Due to this fact and to overcome such limitations, a number of extensions of FSs have been introduced in the literature. By the way, among the most known extensions of FSs, hesitant fuzzy set (HFS) has attracted great attention of many scholars that have been extended to new types and these extensions have been used in many areas such as decision making, aggregation operators, and information measures. Because of such a growth, throughout the present manuscript, we are going to give a thorough and systematic review to the main research results in the field of information measures for HFSs including the distance measures, the similarity measures, and the entropy measures. What seems more considerable in this study is the systematic transformation of the distance measure into the similarity measure and vice versa, and moreover, the two categories of entropy measures including those are derived from the other information measures, and those are based on axiomatic frameworks.

***Keywords*—**Hesitant fuzzy set; Distance measure; Similarity measure; Entropy measure.

I. INTRODUCTION

THESE days there exist a number of extensions of fuzzy sets (FSs) which are known as (i) Atanassov's intuitionistic fuzzy set (IFS) [1, 2] which allows to consider simultaneously the membership degree and the non-membership degree of each element, (ii) Type-2 fuzzy set (T2FS) [3] that incorporates uncertainty in the definition of membership function where a fuzzy set over [0,1] is used to model it, (iii) Interval-valued fuzzy set (IVFS) [4] that assigns to each element a closed subinterval of [0,1] as the membership degree of that element such that the length of the interval may be understood as a measure of the lack of certainty for building the precise membership degree of the element, (iv) Fuzzy multiset (FM) [5] that is based on multiset in which elements can be repeated.

By the way, it is known that whenever the membership degree of an element needs to be established, then the difficulty in such a case is not because of an error margin (as in IFS) or due to some possibility distribution (as in T2FS), but it is because of existing some possible values that make to hesitate about which one would be the right one. To cope with such a challenging situation, Torra [6,7] introduced a new extension of FSs, and called it hesitant fuzzy set (HFS). This concept can be used to model a situation that is very usual in decision making when an expert might consider different degrees of membership of an element, and it is defined as: let X be a reference set. A hesitant fuzzy set (HFS) on X is a function

$$h: X \to \wp([0,1]). \quad (1)$$

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which returns a non-empty subset of values in [0,1]. In the latter definition, the set $\wp([0,1])$ is used to denote the non-empty subset of values in [0,1], and the set of all HFSs on the reference set X is denoted by $\mathbb{HFS}(X)$.

In view of the above definition, Xia and Xu [2] then put forwarded the seminal definition of HFS with an easier mathematical representation: let X be a fixed reference set. A HFS on X is defined in terms of a function from X to a subset of [0,1] which is characterized as:

$$A = \{\langle x, h_A(x)\rangle | x \in X\}, \quad (2)$$

in which $h_A(x)$ is called a hesitant fuzzy element (HFE) that denotes a set of some values in [0,1] and it stands for the possible membership degrees of the element $x \in X$ to the set A. Taking the latter notation into account, one finds that the HFS $A = \cup_{x\in X} \{h_A(x)\}$ is defined based on the set of all HFEs of A.

We organize this article into the following three sections to deal with three different but related issues. Section 2 investigates the review of distance measures for hesitant fuzzy information. Section 3 focuses on the similarity measures for hesitant fuzzy information where on the basis of the relationship between the similarity measure and the distance measure, one can get various formulas to obtain the desired measure. In particular, on the systematic transformation of the distance measure into the similarity measure for HFSs/HFEs and vice versa, we can derive more formulas for the similarity measures of HFSs/HFEs. Section 4 is devoted to the entropy measure as one of the main subject of multiple criteria decision making (MCDM) models with hesitant fuzzy information. On the basis of existing works, we can divide the HFS/HFE entropy measures into two categories: entropy measures derived from the other information measures, and entropy measures which are based on axiomatic frameworks.

II. Distance Measures for Hesitant Fuzzy Sets

Distance measures are fundamentally important in various fields such as decision making, market prediction, and pattern recognition. Such an important role of distance measures in decision-making reveals that they should be investigated thoroughly from different aspects, together with their applicable properties [8]-[11].

In the present section, we review different distance measures that are proposed for HFSs/HFEs, and discuss their issues from different perspectives.

As the pioneer work in this regard, Xu and Xia [12] represented a number of distance measures for HFSs. In the sequel, Peng et al. [13], who discussed on Xu and Xia's [12] distance measures, focused not only on the weight of the individual distance itself, but also on the position weight with respect to the individual distance value. This motivated Peng et al. [13] to propose a generalized hesitant fuzzy synergetic weighted distance measure. Zhou and Li [14] proposed some other distance measures after modifying Xu and Xia's [12] axiomatic definition of distance measure for HFSs. Then, Farhadinia [15] emphasized that the distance measures should fulfil more properties than those considered by Peng et al. [16] and by Xu and Xia [12]. Li et al. [17] showed that Xu and Xia's distance measures do not satisfy some fundamental properties, and gave another version of axiomatic definition of distance measure for HFSs. Moreover, to avoid the encountered drawbacks, Li et al. [17] extended a class of HFEs uniformly by getting a collection of HFEs with the same length.

Notice that all the aforementioned distance measures for HFSs are the value-based measures, that is, they are defined only on the basis of difference between the values of the HFEs, and this is why, the influence of hesitancy index of HFEs has been ignored.

To overcome such a drawback, two aspects have been considered: one is represented by Zhang and Xu [18] who proposed a deviation-based hesitancy index being based on the measure of the pair-wise deviations among possible values of a HFE. The other is that suggested by Li et al.'s [19] who represented a cardinal-based hesitancy index being based on the number of values of a HFE. By comparison of the two latter methods, we find that the calculation of distance measures according to Zhang and Xu's [18] method depends strongly on the procedure of unifying HFEs, that is, the unification is just used for two-by-two HFEs which may make the HFEs at the end of comparison process have different lengths. Meanwhile, Li et al.'s [19] method has unified all the HFEs according to the maximum number of elements from the beginning.

However, a number of methods have some limitations, for example, (i) They need to priorly sort the HFEs which is an extra burden compared with the previous methods; (ii) They are required to add the minimum value or the maximum value extremely which emphasizes the subjectivity of the decision maker, and (iii) It is not usually easy to determine the decision maker's risk altitude or the degree of the decision maker's risk preference.

To overcome such obstacles, Hu et al. [20] presented a series of distance measures for HFEs where the measures are directly calculated from HFEs without judging the decision-makers' risk preference and adding any values into the HFE with the smaller number of elements.

In summarize, a distance measure may fulfil five properties such as the notions of FSs, and their extensions including IFSs, IVFS, and T2FS. Farhadinia [21] modified the axiomatic definition of distance measures for HFSs by taking the following properties into account: Let $A, B, C \in \mathbb{HFS}(X)$. Then d is called a distance measure for HFSs if it possesses the following properties:

a) $0 \le d(A,B) \le 1$;
b) $d(A,B) = d(B,A)$;
c) $d(A,A^c) = 1$ if and only if A is the empty HFS O^* or the full HFS I^*;
d) $d(A,B) = 0$ if and only if $A = B$;
e) If $A \le B \le C$, then $d(A,B) \le d(A,C)$ and $d(B,C) \le d(A,C)$,

where $A^c = \{\langle x, h_{A^c}(x) = \cup_{\gamma \in h_A(x)} \{1-\gamma\}\rangle | x \in X\}$ is the complement set of the HFS A.

III. Similarity Measures for Hesitant Fuzzy Sets

The similarity measure has become an important tool for a variety of different applications ranging from the clustering analysis, pattern recognition to medical diagnosis.

What is remarkable in analysing similarity measures for HFSs is the existing relationships between the axioms for similarity measures and those for distance measures. Indeed, by the help of them, any distance measure formulation can be used to produce its counterpart similarity measure, and vice versa. Due to this close relationship with distance measures, the HFS similarity measures can be naturally applied to many real-world situations where the distance measures of HFSs and their extensions have been applied.

Needless to say that, besides the relationships between the similarity measures and the distance measures, there exist some other kinds of similarity measures that are constructed independently of distance measures, and of course some of them are discussed here.

Similarity measure is indeed obtainable from a distance measure d by taking simply $S(A,B) = Z(d(A,B))$ where Z is a monotone decreasing real function such that $Z(1) = 0$ and $Z(0) = 1$. One of the rules to get a similarity measure from distance measure is the relationship $S_d(A,B) = \frac{Z(d(A,B)) - Z(1)}{Z(0) - Z(1)}$ which defines a similarity measure for the HFSs A and B on the basis of the corresponding distance measure d.

Besides that, Zhang and Xu [18] proposed another generalized formula of hesitant fuzzy similarity measures for HFSs as $S(A,B) = \frac{d(A,B^c)}{d(A,B) + d(A,B^c)}$ where B^c is the complement of HFS B, and $d(A,B)$ is any distance measure for HFSs.

However, similarity measures have attracted a lot of attention in the last decades due to the fact that they can be applied to many areas such as pattern recognition (Li and Cheng [22]), clustering analysis (Yang and Lin [23]), approximate reasoning (Wang et al. [24]), image processing (Pal and King [5]), medical diagnosis (Szmidt and Kacprzyk [25]) and decision making (Xu [26]).

There is a close relationship between the distance measures and the similarity measures. Due to this close relationship with distance measures, the HFS similarity measures can be naturally applied to many real-world situations where the distance measures of HFSs and their extensions have been applied.

On the basis of developed distance measures of HFSs, we can develop some hesitant fuzzy similarity measures by taking the hesitancy indices of HFSs into account:

Let $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle x, h_B(x)\rangle | x \in X\}$ be two HFSs on X. Then S is called a similarity measure for HFSs if it possesses the following properties:

a) $0 \leq S(A,B) \leq 1$ (*Boundary axiom*);
b) $S(A,B) = S(B,A)$ (*Symmetry axiom*);
c) $S(A,A^c) = 0$ if $A = \{\langle x,0\rangle | x \in X\}$ or $A = \{\langle x,1\rangle | x \in X\}$ (*Complementarity axiom*);
d) $S(A,B) = 1$ if and only if $A = B$ (*Reflexivity axiom*).
e) If $A \leq B \leq C$, then $S(A,C) \leq S(A,B)$ and $S(A,C) \leq S(B,C)$.

The following relationship allows us to construct a family of similarity measures for HFSs using a distance between HFSs:

Let $Z: [0,1] \to [0,1]$ be a strictly monotone decreasing real function, and d be a distance between HFSs. Then, for any HFSs A and B on X,

$$S_d(A,B) = \frac{Z(d(A,B)) - Z(1)}{Z(0) - Z(1)} \quad (3)$$

is a similarity measure for HFSs based on the corresponding distance d.

Using the above-mentioned rule, different formulas can be developed to calculate the similarity measure between HFSs using different strictly monotone decreasing functions $Z: [0,1] \to [0,1]$, for instance, (1) $Z(t) = 1 - t$; (2) $Z(t) = \frac{1-t}{1+t}$; (3) $Z(t) = 1 - te^{t-1}$ and (4) $Z(t) = 1 - t^2$.

Zhang and Xu [18] proposed another generalized formula of hesitant fuzzy similarity measures for HFSs by drawing on the idea of similarity measure of IFSs [27]. Indeed, they introduced another kind of similarity measures for HFSs which are obtained by different manners of the above rule. Zhang and Xu [18] defined that

$$S(A,B) = \frac{d(A,B^c)}{d(A,B) + d(A,B^c)} \quad (4)$$

where $d(A,B)$ can be chosen from the set of distance measures defined in the previous section.

What we should note here is that the latter similarity measure S involves both similarity and dis-similarity, that is, it can not only take into account a pure distance between HFSs but also examine whether the compared HFSs are more similar or more dis-similar to each other. Such a desirable characteristic makes it possible to avoid drawing conclusions about strong similarity between HFSs based on the small distances between these two sets [27].

IV. ENTROPY MEASURES FOR HESITANT FUZZY SETS

The notion of entropy for FSs and their extensions allows us to measure the degree of fuzziness, ambiguity, or the uncertainty of a set which returns the amount of difficulty in making a decision whether an element belongs to that set or not.

By the way, in this section, we will discuss about the constructional approaches of the entropy measures for HFEs including the entropy measures which are constructed based on the transforming relationship among the information measures, and those are constructed based on only the axiomatic framework of entropy measures.

Generally, the entropy measures for HFSs are divided into the two conceptual structures: entropy measures based on information measures, and entropy measures based on axiomatic frameworks.

As the first attempt to introduce entropy measures for HFSs, we may refer to the work of Xu and Xia [28]. Then, Farhadinia [21] showed that not only Xu and Xia's [28] method cannot distinguish different HFEs correctly in some situations, but also the formulas cannot construct a variety of entropy measures to quantify the degree of fuzziness of a HFS.

Furthermore, Farhadinia [21] investigated the relationship between the distance measure, the similarity measure and the entropy measure for HFSs based on their axiomatic definitions. After that, Hu et al. [20] developed a series of information measures for HFEs, and subsequently for HFSs, persisting on that these information measures are directly calculated from HFEs without judging the decision makers' risk preference and without needing to add any values into the HFE with the smaller number of elements. Hu et al. [20] claimed that the previous procedures have some limitations, and then they proposed another method of computing information measures for HFEs. By comparing the results of Hu et al.'s [20] method with that of Xu and Xia [28] and Farhadinia [21], it shows that the results of Hu et al. [20] are more in accordance with our intuition. Hu et al. [20] presented another kind of entropy measure that is constructed by taking a hesitant operation and the outranking relation into consideration.

In the sequel, Zhao et al. [29] indicated that Xu and Xia's [28] and Farhadinia's [21] HFE entropy measures have some drawbacks because these entropy formulas are based on only fuzziness of HFEs.

To overcome such a drawback, Zhao et al. [29] redefined the other set of axioms for the entropy measure of HFEs involving the two concepts of uncertainty associated with a HFE that are referred to as the fuzziness and the nonspecificity.

A. *Entropy measures based on information measures*

Xu and Xia [28] put forward some axioms to describe the fuzziness degree of a HFE. They first gave the axiomatic definition of entropy for HFEs as follows:

Let $h_A(x) = \{h_A^{\sigma(j)}(x)\}_{j=1}^{l_x}$ and $h_B(x) = \{h_B^{\sigma(j)}(x)\}_{j=1}^{l_x}$ be two HFEs on X. Then E is called an entropy for HFEs if it possesses the following properties:

a) $0 \leq E(h_A(x)) \leq 1$;

b) $E(h_A(x)) = 0$ if and only if $h_A(x) = O^*$ or $h_A(x) = I^*$;

c) $E(h_A(x)) = 1$ if and only if $h_A^{\sigma(j)} + h_A^{\sigma(l_x-j+1)} = 1$ for $j = 1, \ldots, l_x$;

d) $E(h_A(x)) = E(h_{A^c}(x))$;

e) $E(h_A(x)) \leq E(h_B(x))$, if $h_A^{\sigma(j)} \leq h_B^{\sigma(j)}$ for $h_B^{\sigma(j)} + h_B^{\sigma(l_x-j+1)} \leq 1$ or $h_A^{\sigma(j)} \geq h_B^{\sigma(j)}$ for $h_B^{\sigma(j)} + h_B^{\sigma(l_x-j+1)} \geq 1$ where $j = 1, \ldots, l_x$.

Indeed, the above definition was developed based on the axiomatic definition of FS.

B. *Entropy measures based on distance measures*

Later, Farhadinia [21] pointed out that not only the entropy formulas for HFEs in Xu and Xia's [28] method cannot distinguish different HFEs correctly in some situations, but also the formulas cannot define the entropy for HFSs or construct a variety of entropy measures to quantify the degree of fuzziness of an HFS.

To do this end, Farhadinia [21] gave the axiomatic definition of entropy measure for HFSs as follows: Let A and B be two HFSs on X. Then E_d is called a distance-based entropy for HFSs if it possesses the following properties:

a) $0 \leq E_d(A) \leq 1$;

b) $E_d(A) = 0$ if and only if $A = O^*$ or $A = I^*$;

c) $E_d(A) = 1$ if and only if $A = \{\overline{\tfrac{1}{2}}\}$;

d) $E_d(A) = E_d(A^c)$;

e) If $d(A, \{\overline{\tfrac{1}{2}}\}) \geq d(B, \{\overline{\tfrac{1}{2}}\})$, then $E_d(A) \leq E_d(B)$,

where $\{\overline{\tfrac{1}{2}}\}$ denotes the HFS $\{\overline{\tfrac{1}{2}}\} = \{\langle x, \tfrac{1}{2}\rangle | x \in X\}$.

Subsequently, Farhadinia [21] investigated the relationship between the distance measure, the similarity measure and the entropy measure for HFSs based on their axiomatic definitions.

In order to put forward some formulas, Farhadinia [21] provided a sequence of theorems on how the mentioned information measures for HFSs can be transformed by each other, among them is: let $Z: [0,1] \rightarrow [0,1]$ be a strictly monotone decreasing real function, and d be a distance between HFSs. Then, for any HFS A

$$E_d(A) = \frac{Z(2d(A, \{\overline{\tfrac{1}{2}}\})) - Z(1)}{Z(0) - Z(1)} \tag{5}$$

is an entropy for HFSs based on the corresponding distance d.

Then, Hu et al. [20] developed a series of information measures for HFEs, and subsequently for HFSs, such that these information measures are directly calculated from HFEs without judging the decision-makers' risk preference and without needing to add any values into the HFE with the smaller number of elements.

This task prevents the use of existing techniques where a set with a fewer number of values is extended by adding the same value several times to have the same length as the others have (see Farhadinia et al. [15], [21], [30,31], [32-38], [39-44], Xu and Xia [12]).

However, Hu et al. [20] claimed that such a procedure has three limitations: (1) It will be an extra burden compared to the previous methods if we sort the HFEs; (2) Addition of minimum or maximum value to the shorter HFE causes the emphasizing of the decision maker's subjectivity. (3) It is a hard work to determine the decision maker's risk altitude and the degree of the decision maker's risk preference in a real world problem.

C. *Entropy measures based on similarity measures*

As can be seen in the following, Farhadinia [21] constructed a variety of entropies for HFSs by using various similarity measures for HFSs: let $Z: [0,1] \rightarrow [0,1]$ be a strictly monotone decreasing real function, and S_d be a similarity measure induced by the distance d between HFSs. Then, for any HFS A,

$$E_d(A) = \frac{Z(2Z^{-1}(S_d(A, \{\overline{\tfrac{1}{2}}\}))) - Z(1)}{Z(0) - Z(1)} \tag{6}$$

is an entropy for HFSs.

D. *Entropy measures based on hesitant operations*

There is a common property between a FS and a HFS that is stated as follows: Whenever the membership of any element $x_i \in X$ to the set A has the degree $\mu(x_i) = 0$ or $\mu(x_i) = 1$, then it will be crisp, and whenever $\mu(x_i) = \frac{1}{2}$ for any $x_i \in X$, it means that the set is the most FS. This property motivates Hu et al. [20] to present a formula of entropy measure of HFEs such that it returns the intersection and union of the HFE and its complement as the same as proposed by Shang and Jiang [45] for FSs.

E. *Entropy measures based on fuzziness and non-specificity*

By reviewing Xu and Xia's [28] and Farhadinia's [21] HFE entropy measures, Zhao et al. [29] showed that the existing entropy formulas have some drawbacks because these entropy formulas are based on only fuzziness of HFEs. Zhao et al. [29] represented that for a HFE (respectively, for a HFS) there exist two kinds of uncertainty, fuzziness and nonspecificity. The fuzziness concept of a HFE describes the departure of the HFE from its nearest crisp set, and the nonspecificity concept of a HFE comes from the imprecise knowledge contained in the HFE.

For more clarification about the concept of non-specificity, we consider the situation where a decision organization is asked to characterize the membership degrees of an element x to a set A.

The decision organization describes the membership degrees as the HFE $h(x) = \{0,1\}$. Intuitively, judging based on this HFE alone, we are not sure whether the element x belongs to the set A or not. On the one hand, if we take into account the value 0 as the membership degree, it means that x absolutely does not belong to A. On the other hand, taking the value 1 as the membership degree indicates that x completely belongs to A. Indeed, there is nonspecificity in determining the belonging degree of element of the HFE $h(x) = \{0,1\}$. This fact shows that the definition of an entropy measure for a HFE should involved two types of uncertainty associated with that HFE.

Zhao et al. [29] redefined the other set of axioms for the entropy measure of HFEs involving the two concepts of uncertainty associated with a HFE as follows: let $h_A(x)$ and $h_B(x)$ be two HFEs on X. Then the pair (E_F, E_{NS}) is called a two-tuple entropy measure for HFEs if it possesses the following properties:

a) $0 \leq E_F(h_A(x)) \leq 1$;
b) $E_F(h_A(x)) = 0$ if and only if $h_A(x) = O^*$ or $h_A(x) = I^*$;
c) $E_F(h_A(x)) = 1$ if and only if $h_A(x) = \{\overline{\tfrac{1}{2}}\}$;
d) $E_F(h_A(x)) = E_F(h_{A^c}(x))$;
e) If $h_A^{\sigma(j)}(x) \leq h_B^{\sigma(j)}(x) \leq \frac{1}{2}$ or $h_A^{\sigma(j)}(x) \geq h_B^{\sigma(j)}(x) \geq \frac{1}{2}$, then $E_F(h_A(x)) \leq E_F(h_B(x))$;

and

f) $0 \leq E_{NS}(h_A(x)) \leq 1$;
g) $E_{NS}(h_A(x)) = 0$ if and only if $h_A(x)$ is a singleton, i.e., $h_A(x) = \{\gamma\}$;
h) $E_{NS}(h_A(x)) = 1$ if and only if $h_A(x) = \{0,1\}$;
i) $E_{NS}(h_A(x)) = E_{NS}(h_{A^c}(x))$;
j) If $|h_A^{\sigma(i)}(x) - h_A^{\sigma(j)}(x)| \leq |h_B^{\sigma(i)}(x) - h_B^{\sigma(j)}(x)|$ for any $i, j = 1,2,\ldots,l_x$, then $E_{NS}(h_A(x)) \leq E_{NS}(h_B(x))$.

The two-tuple entropy measure (E_F, E_{NS}) describes how far the HFE is from its closest crisp set together with how non-specific is the information expressed by the HFE.

V. CONCLUSIONS

In this contribution, we tried to give a thorough review to the main research results in the field of information measures for hesitant fuzzy sets (HFSs) including the distance measures, the similarity measures, and the entropy measures.

In terms of other measures, future research on the measure of HFS will include other measure forms, such as compatibility measure, cross entropy measure, etc.

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