About the Double Hierarchy Linguistic Term Set and Its Extensions

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Abstract—Practical decision-making process mainly consists of three aspects: collecting assessment information, processing assessment information, and obtaining the final decision-making result. In this process, the first and most important step is to represent the original meaning of the assessment information of decision-makers (DMs). Considering that the natural languages are more in line with the real thought of DMs than the crisp values, so the research of how to represent qualitative information becomes increasingly popular and attracted many experts. Then, The fuzzy linguistic approach was proposed to deal with the natural language and the concept of computing with words was further defined. Motivated by the fuzzy linguistic approach, the double hierarchy linguistic term set (DHLTS) and its extensions were defined to represent the complex linguistic information, and lots of researches have been done about the DHLTS and its extensions. In this paper, we mainly review the existing research about the DHLTS and its extensions from different angles including decision-making methodologies, preference relations and information measures. Then, we make a short comment about them and discuss some research directions for the future.

Keywords—Double hierarchy linguistic term set; Double hierarchy hesitant fuzzy linguistic term set; Measures; Preference relations; Decision-making methods; Research directions

I. Introduction

PRACTICAL decision-making process mainly consists of three aspects: collecting assessment information, processing assessment information, and obtaining the final decision-making result. In this process, the first and most important step is to represent the original meaning of the assessment information of decision-makers (DMs). Considering that the natural languages are more in line with the real thought of DMs than the crisp values, so the research of how to represent qualitative information becomes increasingly popular and attracted many experts. In this regard, the fuzzy linguistic approach was proposed by Zadeh [1], which can be used to deal with natural languages. Then Zadeh [2] further introduced the concept of Computing with Words (CWW) [2] to operate on linguistic terms, as well as CWW can be explained "*Computing with words is a system of computation in which the objects of computation are words, phrases and propositions drawn from a natural language. The carriers of information are propositions. It is important to note that Computing with words is the only system of computation which offers a capability to compute with information described in a natural language. [2]*" Motivated by the CWW, amounts of linguistic representation models have been developed, such as the 2-tuple linguistic model [3], type-2 fuzzy sets [4], virtual linguistic term model [5], hesitant fuzzy linguistic term set (HFLTS) [6-10], probabilistic linguistic term set (PLTS) [11-14], continuous interval-valued linguistic term set [15], etc. However, sometimes there are lots of flaws when expressing linguistic information using these linguistic representation models mentioned above: Firstly, sometimes we can only utilize some linguistic models to represent one or more basic linguistic terms and cannot express a complex linguistic information using several linguistic labels simultaneously. For example, the 2-tuple linguistic model only can be used to express an uncertain linguistic information by a linguistic label and a real number, and we can only utilize an approximate linguistic label to represent linguistic information by virtual linguistic term model; Secondly, some approximate linguistic labels have no clear meanings such as the virtual linguistic term model; (3) Utilizing real numbers or partial real numbers to represent linguistic information may lose the true meaning of the original linguistic information, such as the type-2 fuzzy set and 2-tuple linguistic model.

In addition to these above gaps, the 2-tuple linguistic model provides inspirations and motivations for scholars to create new complex linguistic models. Motivated by the 2-tuple linguistic structure, Gou et al. [16] defined a new concept named the Double Hierarchy Linguistic Term Set (DHLTS), which is one of the most typical representations and can be used to express the 2-tuple linguistic structure based on two hierarchy linguistic term sets (LTSs). It can be used to express cognitive complex linguistic information accurately with two simple linguistic hierarchies. The first hierarchy LTS is the main linguistic hierarchy and the second hierarchy LTS is the

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linguistic feature or detailed supplementary of each linguistic term included in the first hierarchy LTS [16]. Moreover, to represent uncertain or hesitant fuzzy linguistic information, Gou et al. [16] extended the DHLTS into hesitant fuzzy environment and the concept of double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) was defined.

Because of the flexibility and comprehensiveness of the DHLTS and DHHFLTS, they have attracted increasing attention of experts and scholars. Some decision-making methodologies have been developed under the DHLTS and DHHFLTS environments, such as the double hierarchy hesitant fuzzy linguistic MULTIMOORA (DHHFL-MULTIMOORA) method [16], free DHHFL-TOPSIS method [17], a framework of DHHFLTS [18], unbalanced double hierarchy linguistic TOPSIS method [19], and DHHFL-PROMETHEE method [20]. In addition, Gou et al. [21] defined the concept of double hierarchy hesitant fuzzy linguistic preference relation (DHHFLPR), and respectively researched the additive consistency [21] and multiplicative consistency [22] of the DHHFLPRs and developed a consensus reaching method [23]. Besides, Gou et al. [24] proposed a series of distance and similarity measures for DHHFLTSs and these measures can be regarded as basic tools for decision-making.

Based on the above research results about the double hierarchy linguistic information and double hierarchy hesitant fuzzy linguistic information, this paper mainly makes a comment for DHLTS and its extensions. The study is organized as follows: Section II reviews some basic concepts of DHLTSs and DHHFLTSs. Section III reviewed the decision-making methods with DHLTS and its extensions. Section IV makes a short comment about the existing researches of DHLTS and its extensions. Conclusions are summarized in Section V.

II. DHLTS AND ITS EXTENSIONS

This section mainly introduces the concepts of DHLTSs and DHHFLTSs, as well as two pairs of equivalent transformation functions which can be used to decrease the computation complicity of operations.

A. The concept of DHLTS and DHHFLTS

Definition 1 [16]. Let $S=\{s_t \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be the first hierarchy LTS, $O^t=\{o_k^t \mid k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\}$ be the second hierarchy LTS of s_t. Then, we call

$$S_O=\{s_{t<o_k^t>} \mid t=-\tau,\ldots,-1,0,1,\ldots,\tau;\, k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\} \quad (1)$$

a DHLTS, where $s_{t<o_k^t>}$ is called a double hierarchy linguistic term (DHLT). Especially, almost all second hierarchy LTSs are different in actual situations and we need to express them using different forms. For convenient, however, we can use a unified form $S_O=\{s_{t<o_k>} \mid t=-\tau,\cdots,-1,0,1,\cdots,\tau;\, k=-\varsigma,\cdots,-1,0,1,\cdots,\varsigma\}$ to express them in this paper. For more details and analyses about the DHLTS, please refer to Ref. [16, 23].

Fig. 1 is drawn to show the DHLTS for better understanding.

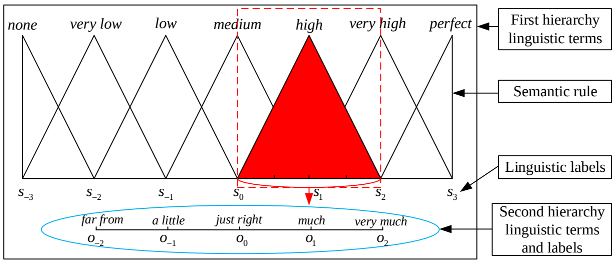

Figure 1. The second hierarchy LTS of the linguistic term s_1

In Fig. 1, we can utilize any one linguistic term included in the second hierarchy LTS O to describe the linguistic term s_1 (*high*). For example, we can use "*very much high*" and "*just right high*" to express the meanings of different degrees of the linguistic term "*high*".

Besides, for different linguistic terms in the first hierarchy LTS, the second hierarchy LTSs of them are different even if the semantic of the second hierarchy LTS is uniform. A figure can be drawn to show these situations, considering that we let $t=3$ and $\varsigma=2$:

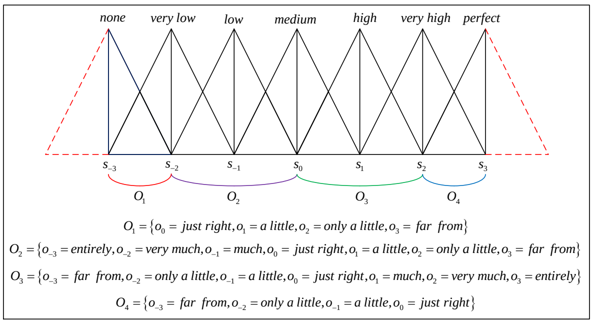

Figure 2. The distributions of the four parts of the second hierarchy LTS

Remark 1. In Fig. 2, four kinds of situations are shown on the basis of different values of t. If $t\geq 0$, then the semantic of the first hierarchy LTS $S=\{s_t \mid t\geq 0\}$ is positive, and then the second hierarchy LTS needs to be given in the ascending order. For example, $s_{1<o_{-1}>}$ (*a little high*) and $s_{1<o_1>}$ (*much high*) are two expressions of s_1, and the degree of the latter one is higher than the former. Similarity, if $t\leq 0$, then the semantic of the first hierarchy LTS $S=\{s_t \mid t\leq 0\}$ is negative, so the second hierarchy LTS needs to be given in the descending order. Especially, we can use LTSs $O=\{o_k \mid k=-\varsigma,\ldots,-1,0\}$ and $O=\{o_k \mid k=0,1,\ldots,\varsigma\}$ describe s_τ and $s_{-\tau}$ respectively considering that the s_τ and $s_{-\tau}$ only have a half of area compared to the complete linguistic terms [16].

Obviously, we can only utilize the linguistic label included in the DHLTS S_O to express a single linguistic term, and cannot express some complex linguistic terms such as "*much high and only a little perfect*" and "*less than only a little fast*". To resolve this issue, Gou et al. [16] extended the DHLTS to the hesitant linguistic environment and defined the concept of DHHFLTS.

Definition 2 [16]. Let $\{x_i \in X \mid i=1,2,\cdots,N\}$ be a fixed set, S_O be a DHLTS. A DHHFLTS on X, H_{S_O}, is in mathematical term of

$$H_{S_O}=\left\{<x_i,h_{S_O}(x_i)>\middle| x_i\in X\right\} \tag{2}$$

where $h_{S_O}(x_i)$ is a set of DHLTs in S_O and can be denoted as:

$$h_{S_O}(x_i)=\left\{s_{\phi_l<o_{\varphi_l}>}(x_i)\middle| s_{\phi_l<o_{\varphi_l}>}\in S_O; l=1,2,\cdots,L;\ \phi_l=-\tau,\cdots,-1,0,1,\cdots,\tau; \varphi_l=-\varsigma,\cdots,-1,0,1,\cdots,\varsigma\right\} \tag{3}$$

with L being the number of DHLTs in $h_{S_O}(x_i)$. $h_{S_O}(x_i)$ denotes the possible degree of the linguistic variable x_i to S_O. $s_{\phi_l<o_{\varphi_l}>}(x_i)$ ($l=1,2,\cdots,L$) in each $h_{S_O}(x_i)$ being the continuous terms in S_O. For convenience, we call $h_{S_O}(x_i)$ a double hierarchy hesitant fuzzy linguistic term (DHHFLE), and all DHLTs in a DHHFLE are ranked in ascending order.

For understanding the DHHFLTS better, the syntax rule of it was given by establishing a context-free grammar $\aleph_{DHH}=\{\dot{V}_N,\dot{V}_T,\dot{I},\dot{P}\}$ as follows [16]:

$\dot{V}_N=\{\langle\text{double hierarchy primary term}\rangle,\langle\text{double hierarchy composite term}\rangle,\langle\text{unary relation}\rangle,\langle\text{binary relation}\rangle,\langle\text{conjunction}\rangle\}$

$\dot{V}_T=\{\text{lessthan; morethan; between; and; } s_{-\tau},s_{1-\tau},\ldots,s_0,\ldots,s_{\tau-1},s_\tau; o_{-\varsigma},o_{1-\varsigma},\ldots,o_0,\ldots,o_{\varsigma-1},o_\varsigma\}$

$\dot{I}=\dot{V}_N$.

For the context-free grammar $\aleph_{DHH}$, the production rules $\dot{P}$ can be defined as:

$\dot{P}=\{\dot{I} ::= \langle\text{double hierarchy primary term}\rangle|\langle\text{double hierarchy composite term}\rangle$

$\langle\text{double hierarchy composite term}\rangle ::= \langle\text{unary relation}\rangle$

$\langle\text{double hierarchy primary term}\rangle \mid \langle\text{binary relation}\rangle\langle\text{double hierarchy primary term}\rangle\langle\text{conjunction}\rangle\langle\text{double hierarchy primary term}\rangle$

$\langle\text{double hierarchy primary term}\rangle ::= s_{-\tau<o_{-\varsigma}>} \mid s_{-\tau<o_{-\varsigma+1}>} \mid \ldots \mid s_{\tau<o_{\varsigma-1}>} \mid s_{\tau<o_\varsigma>}$

$\langle\text{unary relation}\rangle ::= \text{less than} \mid \text{more than}$

$\langle\text{binary relation}\rangle ::= \text{between}$

$\langle\text{conjunction}\rangle ::= \text{and}\}$.

B. *Operations of DHHFLEs*

Before giving the operations of DHHFLEs, motivated by Ref. [25], two pairs of monotonic functions were developed to make mutual transformations between the DHLT (or DHHFLE) and the numerical scale (or hesitant fuzzy element [26]) based on a virtual DHLTS (VDHLTS) $\bar{S}_O$ where the labels are continuous.

Definition 3 [16]. Let $\bar{S}_O=\left\{s_{t<o_k>}\middle| t\in[-\tau,\tau];\ k\in[-\varsigma,\varsigma]\right\}$ be a VDHLTS. Then, a numerical value γ and the subscripts (ϕ,φ) of any a DHLT $s_{\phi<o_\varphi>}$ that expresses the equivalent information to the numerical value γ can be transformed to each other by two monotonic functions f and f^{-1}:

$$f:[-\tau,\tau]\times[-\varsigma,\varsigma]\to[0,1],\ f(\phi,\varphi)=\frac{\varphi+(\tau+\phi)\varsigma}{2\varsigma\tau}=\gamma \tag{4}$$

$$f^{-1}:[0,1]\to[-\tau,\tau]\times[-\varsigma,\varsigma],\ f^{-1}(\gamma_l)=[2\tau\gamma_l-\tau]<o_{\varsigma(2\tau\gamma_l-\tau-[2\tau\gamma_l-\tau])}> \text{ or } [2\tau\gamma_l-\tau]+1<o_{\varsigma((2\tau\gamma_l-\tau-[2\tau\gamma_l-\tau])-1)}> \tag{5}$$

Then, we can introduce the transformation functions F and F^{-1} between the HFLE h_{S_O} and the HFE h_γ:

$$F:\Phi\times\Psi\to\Theta,\ F(h_{S_O})=F\left(\left\{s_{\phi_l<o_{\varphi_l}>}\middle| s_{\phi_l<o_{\varphi_l}>}\in S_O; l=1,\cdots,L;\phi_l\in[-\tau,\tau];\varphi_l\in[-\varsigma,\varsigma]\right\}\right)=\{\gamma_l\mid\gamma_l=f(\phi_l,\varphi_l)\}=h_\gamma \tag{6}$$

$$F^{-1}:\Theta\to\Phi\times\Psi,\ F^{-1}(h_\gamma)=F^{-1}\left(\left\{\gamma_l\middle|\gamma_l\in[0,1]; l=1,2,\cdots,L\right\}\right)=\left\{s_{\phi_l<o_{\varphi_l}>}\mid \phi_l<o_{\varphi_l}>=f^{-1}(\gamma_l)\right\}=h_{S_O} \tag{7}$$

These functions are the foundation of some distance and similarity measures and decision-making methods [16-24]. Based on the above equivalent transformation functions, some operations of DHHFLEs were developed [16] as follows:

Definition 4 [16]. Let $S_O=\left\{s_{t<o_k>}\middle| t=-\tau,\ldots,-1,0,1,\ldots,\tau,\ k=-\varsigma,\ldots,-1,0,1,\ldots,\varsigma\right\}$ be a DHLTS, $h_{S_O}=\left\{s_{\phi_l<o_{\varphi_l}>}\middle| s_{\phi_l<o_{\varphi_l}>}\in S_O; l=1,2,\ldots,L;\phi_l=[-\tau,\tau];\varphi_l=[-\varsigma,\varsigma]\right\}$, $h_{S_{O_1}}=\left\{s^1_{\phi_l<o_{\varphi_l}>}\middle| s^1_{\phi_l<o_{\varphi_l}>}\in S_O; l=1,2,\ldots,L;\phi_l=[-\tau,\tau];\varphi_l=[-\varsigma,\varsigma]\right\}$, and $h_{S_{O_2}}=\left\{s^2_{\phi_l<o_{\varphi_l}>}\middle| s^2_{\phi_l<o_{\varphi_l}>}\in S_O; l=1,2,\ldots,L;\phi_l=[-\tau,\tau];\varphi_l=[-\varsigma,\varsigma]\right\}$ be three DHHFLEs, λ be a real number. Then

(1) $h_{S_{O_1}}\oplus h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F(h_{S_{O_1}}),\eta_2\in F(h_{S_{O_2}})}\{\eta_1+\eta_2-\eta_1\eta_2\}\right)$;

(2) $h_{S_{O_1}}\otimes h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F(h_{S_{O_1}}),\eta_2\in F(h_{S_{O_2}})}\{\eta_1\eta_2\}\right)$;

(3) $\lambda h_{S_O}=F^{-1}\left(\bigcup_{\eta\in F(h_{S_O})}\{1-(1-\eta)^\lambda\}\right)$;

(4) $(h_{S_O})^\lambda=F^{-1}\left(\bigcup_{\eta\in F(h_{S_O})}\{\eta^\lambda\}\right)$;

(5) $h_{S_{O_1}}\ !\ h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F(h_{S_{O_1}}),\eta_2\in F(h_{S_{O_2}})}\{\theta\}\right)$, where

$$\theta=\begin{cases}\dfrac{\eta_1-\eta_2}{1-\eta_2}, & if\ \eta_1\ge\eta_2\ and\ \eta_2\ne 1\\ 0, & otherwise\end{cases};$$

(6) $h_{S_{O_1}}\ \%\ h_{S_{O_2}}=F^{-1}\left(\bigcup_{\eta_1\in F(h_{S_{O_1}}),\eta_2\in F(h_{S_{O_2}})}\{\theta\}\right)$, where

$$\theta = \begin{cases} \frac{\eta_1}{\eta_2}, & if\ \eta_1 \le \eta_2\ and\ \eta_2 \ne 0 \\ 0, & otherwise \end{cases};$$

(7) $\overline{h_{S_O}} = F^{-1}\left(\bigcup_{\eta \in F(h_{S_O})} \{1-\eta\} \right)$;

(8) $h_{S_{O_1}} \cup h_{S_{O_2}} = \left\{ s_{t<o_k>} \middle| s_{t<o_k>} \subset h_{S_{O_1}}\ or\ s_{t<o_k>} \subset h_{S_{O_2}} \right\}$;

(9) $h_{S_{O_1}} \cap h_{S_{O_2}} = \left\{ s_{t<o_k>} \middle| s_{t<o_k>} \subset h_{S_{O_1}}\ and\ s_{t<o_k>} \subset h_{S_{O_2}} \right\}$.

III. DECISION-MAKING METHODS WITH THE DHLTS AND ITS EXTENSIONS

The section mainly reviews the existing research results about the decision-making methods with DHLTS and its extensions.

A. Information measures

Considering that the distance and similarity measures are the basis of decision making with double hierarchy hesitant fuzzy linguistic information, Gou et al. [19] developed a series of distance and similarity measures for DHHFLEs and DHHFLTSs from different angels including the distance and similarity measures with preference information, the distance and similarity measures in continuous cases, the distance and similarity measures between DHHFLTSs with weight information, and the ordered weighted distance and similarity measures between DHHFLTSs.

B. Decision-making methodologies

Under double hierarchy hesitant fuzzy linguistic environment, lots of decision-making methodologies were developed. Firstly, Gou et al. [16] proposed the DHHFL-MULTIMOORA method, and applied it to deal with a practical case about selecting the best city in China by evaluating the implementation status of haze controlling measures. In addition, Montserrat-Adell et al. [17] enhanced the concept of DHHFLTS and defined the concept of free DHHFLTS, and then developed a free DHHFL-TOPSIS method to handle a case study based on tourism management in Barcelona. Moreover, Krishankumar et al. [18] introduced some aggregation operators of DHHFLTSs and proposed a decision-making framework with DHHFLTSs, which was applied to a practical case involving the construction project risk management technique selection. Furthermore, considering that unbalanced semantics may appear in the first and second hierarchy linguistic term sets, the unbalanced double hierarchy linguistic term set (UDHLTS) was proposed by Fu and Liao [19], and then they proposed an unbalanced double hierarchy linguistic TOPSIS (UDHL-TOPSIS) method for multi-expert qualitative decision making problems, and gave an engineering example concerning the green mine selection to illustrate this method. Liu et al. [20] developed a DHHFL-PROMETHEE method and used it to evaluate public-private-partnership's advancement with subjective and objective information from stakeholder perspective.

C. DHHFLPR

Many experts prefer to provide their preferences by making pairwise comparisons between any alternatives. This kind of preferences reflect the relationships between different alternatives intuitively. In this sense, preference relation is one of the popular and effective tools for aiding decision making process.

Based on the DHHFLTS, Gou et al. [21] introduced the concept of DHHFLPR. Before giving the definition of additive DHHFLPR, it is necessary to develop the Addition and Multiplication for DHHFLEs under some conditions:

Definition 5 [21]. Let $S_O = \left\{ s_{t<o_k>} \middle| t = -\tau,\ldots,-1,0,1,\ldots,\tau;\ k = -\varsigma,\ldots,-1,0,1,\ldots,\varsigma \right\}$ be a DHLTS, $h_{S_O} = \left\{ s_{\phi_l<o_{\varphi_l}>} \middle| s_{\phi_l<o_{\varphi_l}>} \in S_O; l = 1,2,\ldots,\#h_{S_O} \right\}$, $h_{S_{O_i}} = \left\{ s_{\phi_l^i<o_{\varphi_l^i}>} \middle| s_{\phi_l^i<o_{\varphi_l^i}>} \in S_O; l = 1,2,\ldots,\#h_{S_O}^i \right\} (i=1,2)$ ($\#h_{S_O}^1 = \#h_{S_O}^2$) be three DHHFLEs, λ be a real number and $0 < \lambda < 1$. Then

(1) $h_{S_{O_1}} \oplus h_{S_{O_2}} = \bigcup_{s_{\phi^1_{\sigma(l)}<o_{\varphi^1_{\sigma(l)}}>} \in h_{S_{O_1}}, s_{\phi^2_{\sigma(l)}<o_{\varphi^2_{\sigma(l)}}>} \in h_{S_{O_2}}} \left\{ s_{\phi^1_{\sigma(l)}+\phi^2_{\sigma(l)}<o_{\varphi^1_{\sigma(l)}+\varphi^2_{\sigma(l)}}>} \right\}$, if $\phi^1_{\sigma(l)} + \phi^2_{\sigma(l)} \le \tau$ and $\varphi^1_{\sigma(l)} + \varphi^1_{\sigma(l)} \le \varsigma$;

(2) $\lambda h_{S_O} = \bigcup_{s_{\phi_{\sigma(l)}<o_{\varphi_{\sigma(l)}}>} \in h_{S_O}} \left\{ s_{\lambda\phi_{\sigma(l)}<o_{\lambda\varphi_{\sigma(l)}}>} \right\}$; $0 < \lambda < 1$;

Specially, if all these three DHHFLEs h_{S_O}, $h_{S_{O_1}}$ and $h_{S_{O_2}}$ only have one DHLT, respectively. Then Definition 2.3 is changed to the operational laws of DHLTs: $\bigoplus_{i=1}^{2} s_{\phi^i<o_{\varphi^i}>} = s_{\phi^1+\phi^2<o_{\varphi^1+\varphi^2}>}$ and $\lambda s_{\phi<o_\varphi>} = s_{\lambda\phi<o_{\lambda\varphi}>}$.

Suppose that $A = \{A_1, A_2, \ldots, A_m\}$ is a fixed set of alternatives. Then a DHHFLPR can be developed:

Definition 6 [21]. A DHHFLPR $\tilde{H}_{S_O}$ is presented by a matrix $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m} \subset A \times A$, where $h_{S_{O_{ij}}} = \left\{ h_{S_{O_{ij}}}^{\sigma(l)} \middle| l = 1,2,\ldots,\#h_{S_{O_{ij}}} \right\}$ ($\#h_{S_{O_{ij}}}$ is the number of DHLT in $h_{S_{O_{ij}}}$, $h_{S_{O_{ij}}}^{\sigma(l)}$ is the l-th DHLT in $h_{S_{O_{ij}}}$) is a DHHFLE, indicating hesitant degrees to which A_i is preferred to A_j. For all $i, j = 1,2,\ldots,m$, $h_{S_{O_{ij}}} (i < j)$ satisfies the conditions:

(1) $h_{S_{O_{ij}}}^{\sigma(l)} + h_{S_{O_{ji}}}^{\sigma(l)} = s_{0<o_0>}$, $h_{S_{O_{ii}}} = s_{0<O_0>}$, and $\#h_{S_{O_{ij}}} = \#h_{S_{O_{ji}}}$;

(2) $h_{S_{O_{ij}}}^{\sigma(l)} < h_{S_{O_{ij}}}^{\sigma(l+1)}$ and $h_{S_{O_{ji}}}^{\sigma(l)} > h_{S_{O_{ji}}}^{\sigma(l+1)}$.

Then, to avoid the occurrence of some self-contradictory situations, it is important to carry out the consistency checking and improving process for each DHHFLPR in GDM process. In this regard, Gou et al. [21] discussed some additive consistency measures for DHHFLPRs.

Definition 7 [21]. Let $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR and $\tilde{H}^N_{S_O} = \left(h^N_{S_{O_{ij}}}\right)_{m\times m}$ be its normalized DHHFLPR, then we call

$\tilde{H}_{S_O}$ an additive consistent DHHFLPR if it satisfies

$$h^N_{S_{O_{ij}}} = h^N_{S_{O_{i\rho}}} \oplus h^N_{S_{O_{\rho j}}} \quad (i, j, \rho = 1,2,\ldots,m; i \neq j) \tag{8}$$

Theorem 1 [21]. Let $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ be a DHHFLPR and $\tilde{H}^N_{S_O} = \left(h^N_{S_{O_{ij}}}\right)_{m\times m}$ be its normalized DHHFLPR. If $\bar{h}^N_{S_{O_{ij}}} = \frac{1}{m}\left(\overset{m}{\underset{\rho=1}{\oplus}}\left(h^N_{S_{O_{i\rho}}} \oplus h^N_{S_{O_{\rho j}}}\right)\right)$ for $i, j, \rho = 1,2,\ldots,m; i \neq j$, then $\tilde{H}_{S_O}$ is an additive consistent DHHFLPR, and $\bar{H}^N_{S_O} = \left(\bar{h}^N_{S_{O_{ij}}}\right)_{m\times m}$ is an additive consistent normalized DHHFLPR.

For the purpose of judging whether a DHHFLPR is of acceptable consistency or not, they defined a consistency index of DHHFLPR and developed a novel method to improve the existing methods for calculating the consistency thresholds, and then gave some novel values of consistency thresholds to check whether a DHHFLPR is of acceptable consistency. They presented two convergent consistency repairing algorithms based on automatic improving method and feedback improving method, respectively, to improve the consistency index of a given DHHFLPR with unacceptable consistency.

Considering that the DHLTS and DHHFLTS can be regarded as the extensions of fuzzy set in linguistic environment and the transformations between each other are equivalent, the multiplicative consistency of fuzzy preference relations and other preference relations were also researched for DHHFLPRs. Gou et al. [22] defined the multiplicative consistency of DHHFLPRs and developed the concept of acceptable multiplicative consistent DHHFLPR

Definition 8 [22]. Given a DHHFLPR $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ and its NDHHFLPR $\tilde{H}^N_{S_O} = \left(h^N_{S_{O_{ij}}}\right)_{m\times m}$ with the parameter $\varepsilon(0 \le \varepsilon \le 1)$. If, for any $i, j, k = 1,2,\ldots,m$,

$$f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right) f\left(h^{N(l)}_{S_{O_{ji}}}\right) = f\left(h^{N(l)}_{S_{O_{ki}}}\right) f\left(h^{N(l)}_{S_{O_{jk}}}\right) f\left(h^{N(l)}_{S_{O_{ij}}}\right) \tag{9}$$

where $h^{N(l)}_{S_{O_{ij}}}$ is the l-th DHLT in the DHHFLE $h^N_{S_{O_{ij}}}$, then $\tilde{H}_{S_O}$ is called a multiplicative consistent DHHFLPR with $\varepsilon(0 \le \varepsilon \le 1)$.

Next, two theorems are developed.

Theorem 2 [22]. Given a DHHFLPR $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ and its NDHHFLPR $\tilde{H}^N_{S_O} = \left(h^N_{S_{O_{ij}}}\right)_{m\times m}$ with the parameter $\varepsilon(0 \le \varepsilon \le 1)$, the following statements are equivalent:

Ⅰ. $\tilde{H}_{S_O}$ is multiplicative consistent.

Ⅱ. $$f\left(h^{N(l)}_{S_{O_{ij}}}\right) = \frac{f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right)}{f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right) + \left(1 - f\left(h^{N(l)}_{S_{O_{ik}}}\right)\right)\left(1 - f\left(h^{N(l)}_{S_{O_{kj}}}\right)\right)}, \quad for\ i, j, k = 1,2,\ldots,m$$

Ⅲ. $$f\left(h^{N(l)}_{S_{O_{ij}}}\right) = \frac{\sqrt[m]{\prod_{k=1}^{m} f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right)}}{\sqrt[m]{\prod_{k=1}^{m} f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right)} + \sqrt[m]{\prod_{k=1}^{m}\left(1 - f\left(h^{N(l)}_{S_{O_{ik}}}\right)\right)\left(1 - f\left(h^{N(l)}_{S_{O_{kj}}}\right)\right)}}, \quad for\ i, j = 1,2,\ldots,m$$

Based on Ⅱ, it can be easily proven that

$$f\left(h^{N(l)}_{S_{O_{ij}}}\right) = \frac{f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right)}{f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right) + \left(1 - f\left(h^{N(l)}_{S_{O_{ik}}}\right)\right)\left(1 - f\left(h^{N(l)}_{S_{O_{kj}}}\right)\right)} = 1 \Bigg/ \left(1 + \left(\frac{1}{f\left(h^{N(l)}_{S_{O_{ik}}}\right)} - 1\right)\left(\frac{1}{f\left(h^{N(l)}_{S_{O_{kj}}}\right)} - 1\right)\right)$$

Therefore, $f\left(h^{N(l)}_{S_{O_{ij}}}\right)$ is an increasing function about $f\left(h^{N(l)}_{S_{O_{ik}}}\right)$ and $f\left(h^{N(l)}_{S_{O_{kj}}}\right)$, and we can obtain

$$0 \le 1 \Bigg/ \left(1 + \left(\frac{1}{f\left(h^{N(l)}_{S_{O_{ik}}}\right)} - 1\right)\left(\frac{1}{f\left(h^{N(l)}_{S_{O_{kj}}}\right)} - 1\right)\right) \le 1$$

which means that Eq. (10) produces a reasonable result because the result of $f\left(h^{N(l)}_{S_{O_{ij}}}\right)$ is included in $[0,1]$.

Theorem 3 [22]. Given a DHHFLPR $\tilde{H}_{S_O} = \left(h_{S_{O_{ij}}}\right)_{m\times m}$ and its NDHHFLPR $\tilde{H}^N_{S_O} = \left(h^N_{S_{O_{ij}}}\right)_{m\times m}$ with the parameter $\varepsilon(0 \le \varepsilon \le 1)$, for $i, j, k = 1,2,\ldots,m$, let

$$\hat{h}^{N(l)}_{S_{O_{ij}}} = f^{-1}\left(\frac{\sqrt[m]{\prod_{k=1}^{m} f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right)}}{\sqrt[m]{\prod_{k=1}^{m} f\left(h^{N(l)}_{S_{O_{ik}}}\right) f\left(h^{N(l)}_{S_{O_{kj}}}\right)} + \sqrt[m]{\prod_{k=1}^{m}\left(1 - f\left(h^{N(l)}_{S_{O_{ik}}}\right)\right)\left(1 - f\left(h^{N(l)}_{S_{O_{kj}}}\right)\right)}}\right) \tag{10}$$

Then, $\hat{H}_{S_O} = \left(\hat{h}^N_{S_{O_{ij}}}\right)_{m\times m}$ is a multiplicative consistent DHHFLPR with ε.

Then Gou et al. [22] proposed a consistency checking method to judge whether a DHHFLPR is of acceptable consistency or not. Additionally, to fully respect and consider the opinions of DMs, a feedback mechanism-based repairing method was developed to improve the consistency of a DHHFLPR.

In addition, Gou et al. [23] developed a consensus reaching method for large-scale group decision making (LSGDM) with DHHFLPRs. To ensure the implementation of consensus reaching process, a similarity degree-based clustering method, a double hierarchy information entropy-based weights-determining method and some consensus measures were developed.

IV. COMMENTS ABOUT THE DHLTS AND ITS EXTENSIONS

According to the above discussions, a short comment about the DHLTS and its extensions can be summarized by analyzing their advantages and disadvantages. Future research directions are provided for researchers and practitioners.

The DHLTS and its extensions mainly have the following advantages:

(1) All elements in a DHLTS are expressed by linguistic labels without any numerical scales, which reflect the semantics of original natural languages to a greater extent.

(2) The second hierarchy LTS is necessary when the set of adverbs of a first hierarchy linguistic term is large. Otherwise, we can add them to the first hierarchy LTS directly.

(3) Each second hierarchy LTS can be regarded as a set of adverbs, which enhances the linguistic representation with richer vocabularies.

(4) Each linguistic term in the first hierarchy LTS has its own second hierarchy LTS, and all of them are usually different.

(5) The DHLTS and its extensions broaden the structure of existing linguistic representation models. In addition, the research of consistency, consensus, and LSGDM enrich the research and application fields of double hierarchy linguistic models.

On the other hand, there still exist some unsolved problems in existing research:

(1) The DHLTS contains much more linguistic information than other existing linguistic representation models. Even though we can decrease the complexity of operations based on the equivalent transformation functions, sometimes the operations are still complex.

(2) The research of consistency measures of DHHFLPR only contains additive consistency and multiplicative consistency. Some novel consistency measures need to be studied. Additionally, it is also necessary to develop some novel consistency checking and repairing methods [27] and not just the automatic improving method and feedback improving methods.

(3) The decision-making methods mentioned above are based on classical methods. In addition to these decision-making methods, some innovative approaches need to be investigated in the future, such as the GLDS (Gained and Lost Dominance Score) method [28, 29] and DNMA (Double Normalization-based Multiple Aggregation) method [30-32].

(4) Along with the research challenges mentioned above, some new and interesting topics about the double hierarchy linguistic model emerge, including novel consistency-driven optimization-based models and consensus reaching methods. In the future, novel consistency measure such as interval consistency can be studied and the research of novel consistency-driven optimization-based models shall be involved.

(5) In addition, in the decision-making process, the assessment information provided by experts or aggregated results may be represented by possible DHHFLEs or DHHFLEs with probability information, and these probabilities are essential to describe the real thoughts of DMs. So we cannot ignore them when representing them directly or aggregating multiple DMs' assessments.

(6) Finally, the self-confident degrees can be used to depict the degrees of confidences that experts have in their own evaluation information. The basic elements DHHFLEs of DHHFLPR are only some linguistic expressions and cannot reflect the self-confident degrees of experts. Considering that there is little research about the DHHFLPRs with self-confident degrees in literature, it is necessary to research the self-confident DHHFLPR in the future.

V. Conclusions

This paper reviewed the basic concepts of DHLTS and its extensions, and introduced the equivalent transformation functions. Then, we reviewed the existing researches of DHLTS and its extensions including the information measures, decision-making methodologies and preference relations. Finally, we made some comments about the DHLTS and its extensions and discussed some research directions in the future.

As novel linguistic representation models, the DHLTS and its extensions can be researched and applied in amounts of fields considering their advantages. The paper provides some insights and research directions for researchers and practitioner who have an interest in this area.

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