



# Network Science and Dynamical Networks

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FOR the simplest form, a network consists of a set of discrete elements called nodes (or vertices), and a set of connections linking these elements called edges (or links), which is shown in Fig. 1 as an illustration. Based on this general definition, it can be concluded that the networks are everywhere and we confront many networks in our daily life [1],[2]. Networks such as Internet, World Wide Web, power grid, social, biological and economical networks have been subject to heavy studies in the last decade.

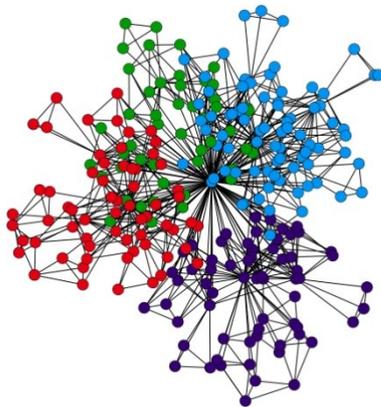


Figure 1. An example network.

Historically, the *Eulerian tour* problem is often cited as the first true proof in the theory of networks. In 1736, Euler studied the Königsberg Bridge problem and in his celebrated solution proved the impossibility of existence of a single path that crosses all seven bridges exactly once each. Later on, many scientists contributed to various developments in network science and in the twentieth century graph theory has developed into a substantial body of knowledge.

*Graph theory* principally has been developed as a pure mathematics discipline, and as such is concerned mainly with the combinatorial properties of static, artificial and rather small constructs. However, although pure graph

theory is well developed, mature and elegant, its results are not especially relevant and explanatory for networks arising from real-world systems. In contrary to the class of networks normally considered in graph theory, the real-world networks are dynamic and evolving in time. They are large and the number of their nodes can vary from thousands to even billion nodes, e.g. World Wide Web. Accordingly, although graph theory donated a rich set of analytical tools, for analyzing and modeling real-world networks a new approach is needed. These new approaches which can be grouped under the name of "network science" initiated by observing the real-world systems, investigating their properties and then trying to construct proper models reproducing them.

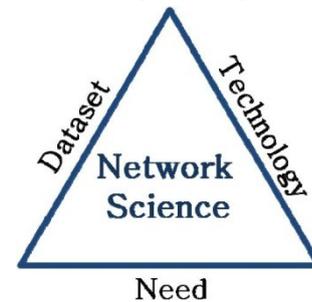


Figure 2. The triangle of driving forces of the network science.

Nowadays, *network science* has become the center of attention and research interests [3]. The volume of publications and investments are sharply increasing and the applications are spreading widely over various disciplines. In fact, this fast growth is due to three sides of a "driving force triangle", as it is shown in Fig. 2, namely; increasing of need, availability of datasets, advent of more efficient computational hardware and new algorithms. This has allowed studying the topology of the interactions in a large variety of systems varying from communication networks to social and biological systems.

**Need:** Researchers have found that many real-world networks from physics to biology, engineering and

sociology have some common structural properties [4]. Studying the properties of such networks could shed light on understanding the underlying phenomena or developing new insights into the system. By studying social networks, for instance, interesting findings have been obtained on spreading of disease and techniques for controlling them [5]. The information extracted from structure and dynamic of the web has enabled researchers to develop efficient algorithms for navigation and search in the web. Investigating biological networks helps us to have better understanding the organization and evolution of their units [6].

**Technology:** The computing technology is subject to daily improvement. The working frequency of CPUs, the transmission speed on the internet, and capacity of storage devices are at least an order of magnitude higher than a decade ago. This technology growth enables us to run more complex algorithms, collect more online information and store huge amounts of data.

**Datasets:** As the last side of the driving force triangle of the network science, we should point to the availability of datasets. Recent years have witnessed a significant increase in the availability of network datasets comprising thousands and sometimes even billions of nodes. In fact, this is a consequence of widespread availability of electronic datasets and more important, the Internet. In traditional approaches, one should spend a lot of effort, i.e. time and money, to collect data and form a network out of it but nowadays, thanks to the internet and communication revolution, one can easily collect data online or download an already collected dataset.

A *directed (undirected) graph* is a graph  $G(V, E)$  where the order of the two nodes in  $E$  is (not) important. In a directed graph,  $e_{ij}$  stands for an edge from  $i$  to  $j$  and is different from  $e_{ji}$  that is an edge from  $j$  to  $i$ . In an undirected graph we have  $e_{ij} = e_{ji}$ . If the edges only represent a connection between two nodes, the graph is called *unweighted* but if a value is attached to each edge, the graph is called *weighted* and is denoted by  $G(V, W)$ , where  $W=\{w_{ij}\}$  is the set of weight values for each edge. Self loops are referred to the edge that connects a node to itself. A simple graph is an undirected graph that contains no self loops. It should be mentioned that in a more general definition of the graph which is not of interest of this report, multiple edges between nodes, different types of nodes and edges are allowed. Graphs with either of these elements are called multigraphs.

Generally, *dynamical network* is a compromise between two extremes where on one side just the whole dynamic of the system is the point of interest and on the other side just the structure of the underlying graph is the target of study. There is the following definition [7],[8].

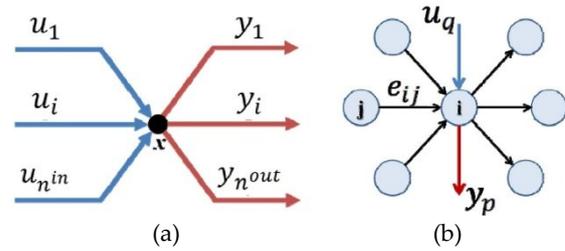
**Definition 1.** [Dynamical network]. A (deterministic) Dynamical Network is composed of

- A directed weighted graph  $G(V, W)$ ;

- A dynamical system on each vertex of the graph. This dynamical system transforms its

1. input signals, corresponding to the incoming edges of the vertex;
2. to its output signals, corresponding to the outgoing edges of the vertex;
3. via its states, corresponding to the vertex itself.

Based on the mentioned notation, the dynamical network (shown as a black circle in Fig. 3(a)) as a large dynamical system has the state vector  $x=(x_1, \dots, x_N)^T$ , the vector of input  $u=(u_1, \dots, u_N)^T$  and the vector of output  $y=(y_1, \dots, y_N)^T$  signals. On the other hand, considering the role of the underlying graph, the dynamical network can be viewed as Fig. 3(b). In this scale, in addition to the external input and output signals (that can be optionally be connected to a node), on an edge from vertex  $j$  to vertex  $i$  travels a signal that is an output signal of the dynamical system at vertex  $j$  and becomes an input signal of the dynamical system at vertex  $i$ .

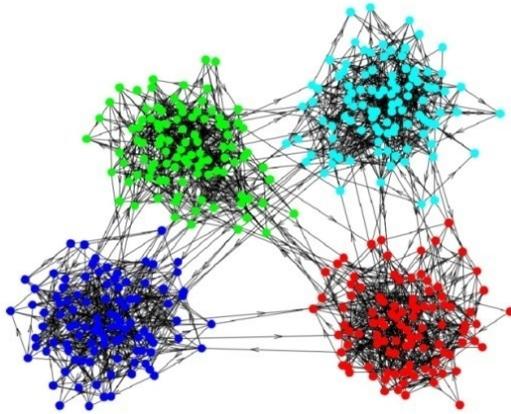


**Figure 3.** (a) State vector, input and output signals for the dynamical network as an integrated large dynamical system, (b) inputs and outputs of a dynamical system that is located on a vertex  $i$  in a dynamical network.

From what has been mentioned, the properties of the dynamical networks can be generally categorized into two main groups namely; structural and dynamical.

Among the structural properties, clusters or communities [9],[10] refer to the phenomenon when nodes of the network can be naturally partitioned into groups such that each group is densely connected internally and the connections between groups are sparse, which is shown in Fig. 4. As a consequence, each node is more likely to be connected with the nodes in the same cluster than that in other clusters. This clustering phenomenon creates divisions in the network, and the understanding of these divisions can lead us to better comprehend not only these objects but also the human behavior or the nature itself. These clusters or communities are important for both the scientific insights on networked interactions and the real-life applications. For example, clustering web clients having the same interests or being geographically close could help achieve a better performance of internet services, identifying groups of purchasers with aligned

interests enables to create more efficient recommendation systems, grouping proteins having similar functions may shed some light on a better understanding of human and natural cellular mechanisms, etc.



**Figure 4.** The illustration of clusters in a network. Here, different clusters are represented by different colors.

### CONCLUSIONS

As humans connect in increasing rates, their relationships, stories, experiences and knowledge interact, which creates a multitude of crowd behaviors that may lead to a better understanding of human relationships. As a consequence, the network science became one of the

major research themes during these last years, as a source of never-ending untapped information serving researchers in many areas of science: psychology, sociology, advertising, statistics, physics, computer science, etc.

The structural properties were in the center of attention from the first days the network analysis emerged. Structural analysis of the real world networks, including degree and load distribution analysis, community detection, and network modeling, helped researchers in wide range of disciplines elucidate many open problems based on some complex network universal concepts such as small world effect, community structures and power law degree distribution. Although the structural properties of complex networks are still receiving attention, recently it has been found that the analysis of the dynamics of the networks is much more interesting.

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